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The Computation of a Continuum of Constrained Equilibria [‡]

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Abstract

The equilibria of a model of an exchange economy with price rigidities are called constrained equilibria. A simplicial algorithm is presented to compute an approximation of a set of constrained equilibria, containing the two so-called trivial equilibria. It is shown that a continuum of constrained equilibria is approximated by the algorithm. Moreover, the convergence properties of the algorithm and the accuracy of the approximate constrained equilibria obtained by the algorithm are discussed. Finally an example is given as an illustration.

Key words. Nonlinear equations, continuum of zero points, economic equilibria, simplicial algorithms.

1 Introduction

Over the last two decades many important and basic existence problems raised in economic theory and game theory have been successfully solved by constructive approaches. Most of the literature on this issue derives from the pioneering work of Scarf (1967), see also Scarf (1973). Scarf introduced an algorithm based on the relationship between Sperner's lemma and Brouwer's fixed point theorem to generate a sequence of adjacent primitive sets in the unit simplex which terminates with an approximate fixed point or equilibrium price vector of an Arrow-Debreu economic model within a finite number of iterations. Later more efficient and sophisticated algorithms have been developed by e.g., Eaves and Saigal (1972), Merrill (1972), and van der Laan and Talman (1979). In these algorithms a sequence of adjacent simplices of a triangulation is generated in order to approximate a fixed point of a continuous function.

There is a clear interaction between the developments in economic theory and game theory on one hand and those in fixed point algorithms on the other. In Shoven and Whalley (1992) and in Kaneko and Yamamoto (1986) simplicial fixed point algorithms are used to compute an approximate equilibrium in a general equilibrium model with taxation and an economic model with indivisible commodities, respectively. By providing constructive equilibrium existence proofs the computational approach often gives more insight into the problems under consideration. Moreover, this approach makes it possible to actually compute an economic equilibrium. An example of the influence of game theory on simplicial fixed point algorithms is given by the paper of van der Laan and Talman (1982). Whereas the algorithms developed before were suitable for a problem defined on the unit simplex or $\mathbf{R}^n$, in their paper an algorithm on the Cartesian product of unit simplices was developed, which is more elegant when computing Nash equilibria of non-cooperative games.

The applications of simplicial fixed point algorithms mentioned above are in principle based on the reformulation of a problem in such a way that the conditions of Kakutani's fixed point theorem or of theorems closely related to this fixed point theorem are satisfied. In some recent work the problem of computing refinements of equilibria and computing equilibria when the conditions of Kakutani's fixed point theorem are not satisfied has been addressed. In Brown, DeMarzo and Eaves (1993) a procedure is given to compute equilibria in a model with incomplete markets. The equilibrium existence problem can be formulated as a fixed point problem for some function, but either the domain of this function is not convex, or the function is not continuous. Using the special structure of the discontinuities they are still able to compute an equilibrium for this model in generic cases. In Talman and Yang (1994) and Yamamoto (1993) the problem of computing a proper Nash equilibrium in a non-cooperative game is considered. When this problem is formulated as a fixed point problem not all fixed points satisfy the properness condition. In order to guarantee that the approximated fixed point indeed corresponds to a proper Nash equilibrium the path of points generated by the algorithm is constructed in such a way that these points satisfy some desired properties.

In this paper we consider an exchange economy with price rigidities and quantity rationing introduced in Drèze (1975). In such an economy, the situation in which demand equals supply is called a constrained equilibrium. We propose a simplicial algorithm with vector labelling which starts from a trivial constrained equilibrium, generates a piecewise linear path contained in a sequence of adjacent simplices of varying dimension and terminates with another trivial constrained equilibrium within a finite number of iterations. We demonstrate that every point on the path generated by the algorithm yields an approximate constrained equilibrium. The algorithm differs from other simplicial algorithms in the sense that a whole path or set of constrained equilibria is generated. This allows the method to find all kinds of constrained equilibria. Moreover, the excess demand is allowed to be a correspondence instead of a function. To do this, we have to deal with specific degeneracy problems. For some work to compute one kind of constrained equilibria given continuous excess demand functions, we refer to van der Laan (1982) and Cornielje and van der Laan (1986).

The rest of the paper is organized as follows. In Section 2 we introduce the model of an exchange economy with price rigidities and rationing. In Section 3 we present the steps of the algorithm in detail and prove the convergence of the algorithm. In Section 4 we analyze the accuracy of the approximate solutions obtained by the algorithm and show that a continuum of constrained equilibria is approximated. Finally, an illustration of the algorithm can be found in Section 5.

2 A Model of an Economy with Price Rigidities

In this section a model of an exchange economy with price rigidities and a rationing system is presented and the equilibrium existence problem for such an economy will be formulated as a fixed point problem. An exchange economy with price rigidities is defined by $E = \left(\{X^i, \succeq^i, w^i\}_{i=1}^m, P\right)$. There are m consumers indexed by $i = 1, \ldots, m$ and n commodities indexed by $j = 1, \ldots, n$. For $i = 1, \ldots, m$, consumer i is characterized by a consumption set X^i, a preference ordering $\succeq^i$ on X^i, and a vector of initial endowments w^i. The set of admissible prices is denoted by P. Since the set of admissible prices is allowed to be restricted, a Walrasian equilibrium price vector at which demand equals supply for every commodity does not necessarily exist. In case of excess demands or excess supplies, rationing can be used to obtain a situation where the excess demands on all markets are zero. A rationing scheme gives lower and upper bounds on the excess demands of all consumers. The description of the economy is extended by a rationing system which is described by functions $\hat{l}$ and $\hat{L}$ following Weddepohl (1987). Both $\hat{l}$ and $\hat{L}$ have mn components and describe rationing schemes for each of the consumers on their excess supplies and on their excess demands, respectively, being permitted in the economy. For $i = 1, \ldots, m$ and $j = 1, \ldots, n$ component $(i-1)\,n+j$ of $\hat{l}$ is denoted by $\hat{l}^i_j$ and component $(i-1)\,n+j$ of $\hat{L}$ is denoted by $\hat{L}^i_j$. In the following, for $k \in \mathbf{N}$, I_k denotes the set of integers $\{1, \ldots, k\}$, C^k denotes the k-dimensional unit cube, so $C^k = \left\{q \in \mathbf{R}^k \mid \forall j \in I_k,\ 0 \leq q_j \leq 1\right\}$, 0^k denotes the vector in $\mathbf{R}^k$ for which each component is equal to 0, 1^k denotes the vector in $\mathbf{R}^k$ for which each component is equal to 1, e^k denotes the k-th unit vector in $\mathbf{R}^n$, and E^k denotes the $k \times k$ identity matrix. In the notation e^k the parameter n is omitted, since the dimension of e^k will always equal the number of commodities, n. If $x, y \in \mathbf{R}^k$ then $x \geq y$ means $x_j \geq y_j$, $\forall j \in I_k$, $x > y$ means $x \geq y$ and $\exists j \in I_k$ such that $x_j > y_j$, and $x \gg y$ means $x_j > y_j$, $\forall j \in I_k$. The set $\left\{x \in \mathbf{R}^k \mid x \gg 0\right\}$ is denoted by $\mathbf{R}^k_{++}$. If S is a subset of $\mathbf{R}^k$ then $\mathrm{Int}\,(S)$ denotes the interior of S in $\mathbf{R}^k$. With respect to the economy E and the rationing system $(\hat{l}, \hat{L})$ the following assumptions are made:

A1. For every $i \in I_m$, X^i is a convex, closed, non-empty subset of $\mathbf{R}^n$, $X^i \subset \mathbf{R}^n_+$, and $X^i + \mathbf{R}^n_+ \subset X^i$.

A2. For every $i \in I_m$ the preference ordering $\succeq^i$ on X^i is transitive, complete, continuous, strongly monotonic, and convex.

A3. For every $i \in I_m$ the initial endowments w^i are an element of $\mathrm{Int}\,(X^i)$.

A4. The set of admissible prices is equal to

$$P = \left\{p \in \mathbf{R}^n_+ \,\middle|\, \underline{p}_j \leq p_j \leq \bar{p}_j,\ \forall j \in I_n\right\},$$

for given $\underline{p}, \bar{p} \in \mathbf{R}^n_{++}$ where $\underline{p}_j \leq \bar{p}_j$, for all $j \in I_n$.

A5. The function $(\hat{l}, \hat{L}) : C^n \to -\mathbf{R}^{mn}_+ \times \mathbf{R}^{mn}_+$ specifying the rationing system is continuous on C^n and satisfies for every $i \in I_m$, $j \in I_n$, and $q \in C^n$,

$$\begin{array}{ll}
\hat{l}^i_j(q) = \hat{l}^i_j(r), \text{ if } r \in C^n \text{ and } q_j = r_j, & \hat{L}^i_j(q) = \hat{L}^i_j(r), \text{ if } r \in C^n \text{ and } q_j = r_j, \\
\hat{l}^i_j(q) = 0, \text{ if } q_j = 0, & \hat{L}^i_j(q) > \sum_{h \neq i} w^h_j, \text{ if } q_j \leq \tfrac{2}{3}, \\
\hat{l}^i_j(q) < -w^i_j, \text{ if } q_j \geq \tfrac{1}{3}, & \hat{L}^i_j(q) = 0, \text{ if } q_j = 1.
\end{array}$$

As has been shown in Debreu (1959) Assumptions A1 and A2 imply that it is possible to represent the preferences of consumer $i \in I_m$ by a continuous utility function u^i from X^i into $\mathbf{R}$. The rationing scheme $(l, L) \in -\mathbf{R}^{mn}_+ \times \mathbf{R}^{mn}_+$ is permitted in the economy if there exists $q \in C^n$ such that $(l, L) = (\hat{l}(q), \hat{L}(q))$. The constrained budget set of consumer

$i \in I_m$ at price $p \in P$ and his individual rationing scheme $(l^i, L^i) \in -\mathbf{R}^n_+ \times \mathbf{R}^n_+$, is denoted by $B^i(p, l^i, L^i)$ and given by

$$B^i(p, l^i, L^i) = \left\{x^i \in X^i \,\middle|\, p \cdot x^i \leq p \cdot w^i,\ l^i \leq x^i - w^i \leq L^i\right\}.$$

The demand of consumer $i \in I_m$ at price $p \in P$ and his individual rationing scheme $(l^i, L^i) \in -\mathbf{R}^n_+ \times \mathbf{R}^n_+$, is denoted by $\delta^i(p, l^i, L^i)$ and given by

$$\delta^i(p, l^i, L^i) = \left\{x^i \in B^i\left(p, l^i, L^i\right) \,\middle|\, u^i(x^i) = \max_{y^i \in B^i(p, l^i, L^i)} u^i(y^i)\right\}.$$

The following definition of a constrained equilibrium is similar to the one given in Drèze (1975).

Definition 2.1 (Constrained Equilibrium)
A constrained equilibrium of the economy $E = (\{X^i, \succeq^i, w^i\}_{i=1}^m, P)$ with rationing system $(\hat{l}, \hat{L})$ is an element $(x^{*1}, \ldots, x^{*m}, l^*, L^*, p^*)$ of the set $\prod_{i=1}^m X^i \times -\mathbf{R}^{mn}_+ \times \mathbf{R}^{mn}_+ \times P$ such that

1. $\forall i \in I_m : x^{*i} \in \delta^i\left(p^*, l^{*i}, L^{*i}\right)$;
2. $\sum_{i=1}^m x^{*i} - \sum_{i=1}^m w^i = 0^n$;
3. $\forall j \in I_n : p^*_j < \bar{p}_j$ implies $L^{*i}_j > x^{*i}_j - w^i_j$, $\forall i \in I_m$, and $p^*_j > \underline{p}_j$ implies $l^{*i}_j < x^{*i}_j - w^i_j$, $\forall i \in I_m$;
4. $\exists q^* \in C^n$ such that $(l^*, L^*) = (\hat{l}(q^*), \hat{L}(q^*))$.

Define components $j \in I_n$ of the function $\hat{p} : C^n \to P$ by

$$\hat{p}_j(q) = \max\left\{\underline{p}_j, \min\left\{(2 - 3q_j)\,\underline{p}_j + (3q_j - 1)\,\bar{p}_j, \bar{p}_j\right\}\right\},\ \forall q \in C^n.$$

It can be shown, like in Herings (1992), that if only prices and rationing schemes are considered which are in the image set of the function $(\hat{p}, \hat{l}, \hat{L}) : C^n \to P \times -\mathbf{R}^{mn}_+ \times \mathbf{R}^{mn}_+$ then all constrained equilibria satisfying Definition 2.1 are obtained. Define the excess demand correspondence $\zeta : C^n \to \mathbf{R}^n$ by

$$\zeta(q) = \sum_{i=1}^m \delta^i\left(\hat{p}(q), \hat{l}^i(q), \hat{L}^i(q)\right) - \sum_{i=1}^m \{w^i\},\ \forall q \in C^n.$$

Then it holds that all zero points of ζ correspond with constrained equilibria and vice versa. More precisely, when $0^n \in \zeta(q^*)$ for some $q^* \in C^n$ then there exists an $x^{*i} \in \delta^i(\hat{p}(q^*), \hat{l}^i(q^*), \hat{L}^i(q^*))$ for every $i \in I_m$ such that $(x^{*1}, \ldots, x^{*m}, \hat{l}(q^*), \hat{L}(q^*), \hat{p}(q^*))$ is a constrained equilibrium. For every $j \in I_n$ the variable $q^*_j \in [0, 1]$ determines the state of market j. In case $0 \leq q^*_j \leq \frac{1}{3}$ there is no binding demand rationing on market j and $\hat{p}_j(q^*) = \underline{p}_j$.

When $\frac{1}{3} \leq q_j^* \leq \frac{2}{3}$ there is no binding rationing on market j and $\underline{p}_j \leq \hat{p}_j(q^*) \leq \bar{p}_j$. When $\frac{2}{3} \leq q_j^* \leq 1$ there is no binding supply rationing on market j and $\hat{p}_j(q^*) = \bar{p}_j$. Using the results in Herings (1992) the following theorem can be shown.

Theorem 2.2
Let be given the economy $E = (\{X^i, \succeq^i, w^i\}_{i=1}^m, P)$ with rationing system $(\hat{l}, \hat{L})$ and let the Assumptions A1, A2, A3, A4, and A5 be satisfied. Then the correspondence $\zeta : C^n \rightarrow \mathbf{R}^n$ satisfies the following conditions:

1. *ζ is a non empty-valued, convex-valued, upper semi-continuous correspondence such that $\cup_{q \in C^n} \zeta(q)$ is bounded;*
2. *$\forall q \in C^n, \ \forall z \in \zeta(q), \ \forall j \in I_n, \ z_j \geq 0$ if $q_j = 0$;*
3. *$\forall q \in C^n, \ \forall z \in \zeta(q), \ \forall j \in I_n, \ z_j \leq 0$ if $q_j = 1$;*
4. *$\forall q \in C^n, \ \forall z \in \zeta(q), \ p(q) \cdot z = 0$.*

Using Conditions 2 and 4 it is not difficult to show that $q = 0^n$ corresponds with a constrained equilibrium where all the excess supply is rationed. From Conditions 3 and 4 it is easily seen that $q = 1^n$ corresponds with a constrained equilibrium where all the excess demand is rationed. In the following section an algorithm is presented which yields a piecewise linear path of points in C^n with end points 0^n and 1^n such that every point on the path corresponds with an approximate constrained equilibrium. This result will be used to show that there exists a continuum of points q in C^n containing 0^n and 1^n and satisfying $0^n \in \zeta(q)$.

3 The Algorithm

In this section we introduce a simplicial variable dimension algorithm which will be used to approximate and prove the existence of a continuum of constrained equilibria. More generally, the algorithm can be used to compute a set of zero points for any correspondence $\zeta : C^n \rightarrow \mathbf{R}^n$ satisfying the following condition.

Condition B The correspondence $\zeta : C^n \rightarrow \mathbf{R}^n$ satisfies:

1. ζ is a non empty-valued, convex-valued, upper semi-continuous correspondence such that $\cup_{q \in C^n} \zeta(q)$ is bounded;
2. $\forall q \in C^n, \ \exists z \in \zeta(q)$ such that for every $j \in I_n$ with $q_j = 0, \ z_j \geq 0$, and for every $j \in I_n$ with $q_j = 1, \ z_j \leq 0$;

3. $\forall q \in C^n$, $\forall z \in \zeta(q)$, $\exists p \in \mathbf{R}^n_{++}$ such that $p \cdot z = 0$.

Notice that by Theorem 2.2 the excess demand correspondence of an economy with price rigidities satisfies Condition B. More precisely, Condition B.1 is the same as Property 1 of Theorem 2.2, Condition B.2 is weaker than Properties 2 and 3 of Theorem 2.2, and Condition B.3 is weaker than Property 4 of Theorem 2.2.

Some preliminaries are given first. A vector $s \in \mathbf{R}^n$ of which each component is -1, 0, or $+1$ is called a sign vector. For a sign vector $s \in \mathbf{R}^n$ define the sets $I^-(s) = \{j \in I_n \mid s_j = -1\}$, $I^0(s) = \{j \in I_n \mid s_j = 0\}$, and $I^+(s) = \{j \in I_n \mid s_j = +1\}$. Moreover, the subset $A(s)$ of C^n is defined by

$$A(s) = \{x \in C^n \mid x_j = 0 \text{ if } s_j = +1,\ x_j = 1 \text{ if } s_j = -1\}.$$

It is easy to see that the dimension of $A(s)$ is equal to $|I^0(s)|$. Moreover, if $|I^0(s)| = t$ then there exists a sequence $i_s^1 < \cdots < i_s^{n-t}$ such that $I^-(s) \cup I^+(s) = \{i_s^1, \ldots, i_s^{n-t}\}$. Notice that the set $A(0^n)$ equals the set C^n.

For given $t \in \mathbf{N}$, a t-dimensional simplex or t-simplex, denoted by σ, is defined as the convex hull of $t+1$ affinely independent points $x^1, \ldots, x^{t+1}$ of $\mathbf{R}^n$. We usually write $\sigma = \sigma(x^1, \ldots, x^{t+1})$ and call $x^1, \ldots, x^{t+1}$ the vertices of σ. A $(t-1)$-simplex being the convex hull of t vertices of $\sigma(x^1, \ldots, x^{t+1})$ is said to be a facet of σ. The facet $\tau(x^1, \ldots, x^{i-1}, x^{i+1}, \ldots, x^{t+1})$ is called the facet of a t-simplex σ opposite to the vertex x^i. For k, $0 \leq k \leq t$, a k-simplex being the convex hull of $k+1$ vertices of σ is said to be a k-face or face of σ. A finite collection $\mathcal{T}$ of n-simplices is a simplicial subdivision or triangulation of C^n if

1. C^n is the union of all simplices in $\mathcal{T}$;

2. the intersection of any two simplices in $\mathcal{T}$ is either empty or a common face of both.

It can be shown that for any sign vector s the collection of t-faces of the simplices in $\mathcal{T}$ lying in the set $A(s)$ induces a triangulation of $A(s)$. Since $\mathcal{T}$ is finite and C^n is compact, each facet τ of a t-simplex $\sigma \in A(s)$ either lies in the relative boundary of $A(s)$ and is only a facet of σ or is a facet of exactly one other t-simplex in $A(s)$. The mesh size of a triangulation $\mathcal{T}$ of C^n is defined by $\text{mesh}(\mathcal{T}) = \max\{\|x - y\|_\infty \mid x, y \in \sigma,\ \sigma \in \mathcal{T}\}$. An example of a triangulation of C^n having an arbitrarily chosen mesh size and that can easily be implemented is the K-triangulation proposed in Freudenthal (1942) and will be described in Section 5.

Let $\zeta : C^n \to \mathbf{R}^n$ be a correspondence satisfying Condition B. A function $Z : C^n \to \mathbf{R}^n$ is called a piecewise linear approximation of ζ with respect to a given triangulation $\mathcal{T}$ of C^n if for each vertex x of any $\sigma \in \mathcal{T}$, $Z(x) \in \zeta(x)$ and Z is affine on each simplex of $\mathcal{T}$.

Moreover, in the following Z will be chosen such that the following boundary condition is satisfied.

Condition C Let $\mathcal{T}$ be a triangulation of C^n. The piecewise linear approximation $Z : C^n \to \mathbf{R}^n$ of ζ with respect to $\mathcal{T}$ is such that for each vertex x in the triangulation $\mathcal{T}$ it holds that $Z_j(x) \geq 0$ if $x_j = 0$, and $Z_j(x) \leq 0$ if $x_j = 1$.

Notice that it is always possible to let a piecewise linear approximation Z of ζ with respect to a triangulation $\mathcal{T}$ of C^n satisfy Condition C if ζ satisfies Condition B.

Let $\sigma(x^1, \ldots, x^{t+1})$ be a t-simplex in C^n and let s be a sign vector with $|I^0(s)| = t$. Consider solutions $(\lambda_1, \ldots, \lambda_{t+1}, \mu_1, \ldots, \mu_{n-t}, \beta) \in \mathbf{R}^{n+2}$ of the following system of equations:

$$\sum_{j=1}^{t+1} \lambda_j \begin{pmatrix} 1 \\ Z(x^j) \end{pmatrix} + \sum_{j=1}^{n-t} \mu_j \begin{pmatrix} 0 \\ s_{i^j_s} e^{i^j_s} \end{pmatrix} + \beta \begin{pmatrix} 0 \\ -1^n \end{pmatrix} = \begin{pmatrix} 1 \\ 0^n \end{pmatrix}. \tag{1}$$

If $\lambda_j \geq 0$, $\forall j \in I_{t+1}$, and $\mu_j \geq 0$, $\forall j \in I_{n-t}$, then $(\lambda_1, \ldots, \lambda_{t+1}, \mu_1, \ldots, \mu_{n-t}, \beta)$ is called an admissible solution to (1). For an admissible solution to (1) corresponding with a simplex $\sigma(x^1, \ldots, x^{t+1})$ in $A(s)$ it holds that $0 \leq Z_j(q) \leq \beta$ if $q_j = 0$, $Z_j(q) = \beta$ if $0 < q_j < 1$, and $0 \geq Z_j(q) \geq \beta$ if $q_j = 1$, where $q = \sum_{j=1}^{t+1} \lambda_j x^j \in \sigma$. As will be shown in the next section if mesh($\mathcal{T}$) is small enough then β will be close to zero. However, despite of Condition B.3 it is impossible to guarantee that $\beta = 0$ and so $Z(q) = 0^n$. This is caused by the fact that Condition B.3 is not necessarily satisfied for a piecewise linear approximation Z of ζ. Next it will be shown that there exists a piecewise linear path in C^n from 0^n to 1^n such that for every point q on the path there is a sign vector s, a t-simplex $\sigma(x^1, \ldots, x^{t+1})$ in $A(s)$ with $t = |I^0(s)|$, and an admissible solution $(\lambda_1, \ldots, \lambda_{t+1}, \mu_1, \ldots, \mu_{n-t}, \beta)$ to the system in (1), satisfying $\sum_{j=1}^{t+1} \lambda_j x^j = q$. In the next section it will be shown that any point q on the path indeed corresponds to an approximate zero point of ζ.

In order to prove the existence of the piecewise linear path with the given properties, no non-degeneracy assumptions will be made. The proof is inspired by Todd (1976) and Wright (1981), where algorithms are presented to compute a zero point of a correspondence $f : \mathbf{R}^n \to \mathbf{R}^n$. In their work lexicographic pivot rules are used in order to circumvent degeneracy. Notice that an admissible solution to (1) is said to be degenerate if at least two of the variables λ_j, $j \in I_{t+1}$, and μ_j, $j \in I_{n-t}$, equal zero. For any piecewise linear approximation Z satisfying condition C of a correspondence ζ satisfying condition B with respect to any triangulation $\mathcal{T}$ of C^n it holds that $Z(0^n) = 0^n$ and $Z(1^n) = 0^n$. Hence $\lambda_1 = 1$, $\mu_j = 0$, $\forall j \in I_n$, and $\beta = 0$ is a degenerate admissible solution to (1) corresponding with the 0-simplex $\sigma(0^n)$ and any sign vector s with $|I^0(s)| = 0$. A similar completely degenerate admissible solution exists corresponding with the 0-simplex $\sigma(1^n)$. So the usual non-degeneracy assumption that there exist no degenerate admissible solutions makes no sense for the problem under consideration. A row vector $x \in \mathbf{R}^n$ is lexicographically positive if $x \neq 0^{n^\top}$ and its first nonzero entry is positive. A matrix A is semi-lexicopositive if each except possibly the last row is lexicographically positive.

For a $(t-1)$-simplex $\tau(x^1, \ldots, x^t)$ and a sign vector s with $|I^0(s)| = t$, the $(n+1)\times(n+1)$ matrix $A_{s,\tau}$ is defined by

$$A_{s,\tau} = \begin{pmatrix} 1 & \cdots & 1 & 0 & \cdots & 0 & 0 \\ Z(x^1) & & Z(x^t) & s_{i_s^1} e^{i_s^1} & & s_{i_s^{n-t}} e^{i_s^{n-t}} & -1^n \end{pmatrix}.$$

Definition 3.1

Let τ be a $(t-1)$-simplex of C^n and let s be a sign vector with $|I^0(s)| = t$. Then τ is s-complete if $A_{s,\tau}^{-1}$ exists and is semi-lexicopositive.

The first column of $A_{s,\tau}^{-1}$ corresponds with an admissible solution to (1) for a t-simplex σ being the convex hull of τ and some vertex $x^{t+1} \in C^n$, and where $\lambda_{t+1} = 0$. We will now construct a finite sequence of simplices starting with the simplex $\{0^n\}$ and terminating with the simplex $\{1^n\}$ satisfying that for every $(t-1)$-simplex τ in the sequence there exists a sign vector s with $t = |I^0(s)|$ such that τ is an s-complete simplex in $A(s)$. Moreover, any two subsequent simplices in the sequence either are facets of the same simplex or one is a facet of the other. We first show that $\tau(0^n)$ and $\tau(1^n)$ are each s-complete simplices in $A(s)$ with respect to some unique sign vector s with $|I^0(s)| = 1$.

Lemma 3.2

Let the sign vector $s \in \mathbf{R}^n$ be such that $s_j = +1$, $\forall j \in I_{n-1}$, and $s_n = 0$. Then the 0-simplex $\tau(0^n)$ is an s-complete simplex in $A(s)$ but is not an $\tilde{s}$-complete simplex in $A(\tilde{s})$ for any other sign vector $\tilde{s} \in \mathbf{R}^n$.

Proof

Suppose that $\tau(0^n)$ is $\tilde{s}$-complete simplex in $A(\tilde{s})$. Condition B.3 and Condition C imply that $Z(0^n) = 0^n$. Since τ is 0-dimensional and τ lies in $A(\tilde{s})$ it has to hold that $|I^0(\tilde{s})| = 1$ and $I^-(\tilde{s}) = \emptyset$. Then $A_{\tilde{s},\tau}$ is given by

$$A_{\tilde{s},\tau} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0^n & e^{i_{\tilde{s}}^1} & & e^{i_{\tilde{s}}^{n-1}} & -1^n \end{pmatrix},$$

where $i_{\tilde{s}}^1 < i_{\tilde{s}}^2 < \cdots < i_{\tilde{s}}^{n-1}$. Define $i_{\tilde{s}}^0 = 0$ and $i_{\tilde{s}}^n = n+1$. Let j be the unique element in the set $I_n \setminus I^+(\tilde{s})$ and let $k \in I_n$ be such that $i_{\tilde{s}}^{k-1} < j < i_{\tilde{s}}^k$. Then

$$A_{\tilde{s},\tau}^{-1} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0^n & e^{i_{\tilde{s}}^1} & & e^{i_{\tilde{s}}^{k-1}} & -1^n & e^{i_{\tilde{s}}^k - 1} & & e^{i_{\tilde{s}}^{n-1} - 1} \end{pmatrix}.$$

It is clear that this matrix is semi-lexicopositive if and only if $k = n$. Q.E.D.

Lemma 3.3

Let the sign vector $s \in \mathbf{R}^n$ be such that $s_1 = 0$ and $s_j = -1$, $\forall j \in I_n \setminus \{1\}$. Then the 0-simplex $\tau(1^n)$ is an s-complete simplex in $A(s)$ but is not an $\tilde{s}$-complete simplex in $A(\tilde{s})$ for any other sign vector $\tilde{s}$.

Proof

Suppose that $\tau(1^n)$ is an $\tilde{s}$-complete simplex in $A(\tilde{s})$. Condition B.3 and Condition C imply that $Z(1^n) = 0^n$. Since τ is 0-dimensional and since τ lies in $A(\tilde{s})$ it holds that $I^0(\tilde{s}) = 1$ and $I^+(\tilde{s}) = \emptyset$. Hence $A_{\tilde{s},\tau}$ is given by

$$A_{\tilde{s},\tau} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0^n & -e^{i^1_{\tilde{s}}} & & -e^{i^{n-1}_{\tilde{s}}} & -1^n \end{pmatrix},$$

where $i^1_{\tilde{s}} < i^2_{\tilde{s}} < \cdots < i^{n-1}_{\tilde{s}}$. Define $i^0_{\tilde{s}} = 0$ and $i^n_{\tilde{s}} = n + 1$. Let j be the unique element in the set $I_n \setminus I^-(\tilde{s})$ and let $k \in I_n$ be such that $i^{k-1}_{\tilde{s}} < j < i^k_{\tilde{s}}$. Then it is easily verified that

$$A^{-1}_{\tilde{s},\tau} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0^n & -e^{i^1_{\tilde{s}}} & & -e^{i^{k-1}_{\tilde{s}}} & \begin{matrix} 1^{n-1} \\ -1 \end{matrix} & -e^{i^k_{\tilde{s}}-1} & & -e^{i^{n-1}_{\tilde{s}}-1} \end{pmatrix}.$$

Clearly, this matrix is semi-lexicopositive if and only if $k = 1$.

Q.E.D.

The following lemma is well-known in linear programming theory, see for example Murty (1983, p. 232). It will be very useful in proving Lemma 3.5 and Lemma 3.6.

Lemma 3.4

Let $A = (A_{\cdot 1}, \ldots, A_{\cdot n+1})$ be a regular $(n+1) \times (n+1)$ matrix and let $x \in \mathbf{R}^{n+1}$. Let $k \in I_{n+1}$ and let $\tilde{A} = (A_{\cdot 1}, \ldots, A_{\cdot k-1}, x, A_{\cdot k+1}, \ldots, A_{\cdot n+1})$. Then either $(A^{-1}x)_k = 0$ and $\tilde{A}$ is singular, or $(A^{-1}x)_k \neq 0$ and

$$\tilde{A}^{-1} = \begin{pmatrix} (A^{-1})_{1\cdot} - \frac{(A^{-1}x)_1}{(A^{-1}x)_k}(A^{-1})_{k\cdot} \\ \vdots \\ (A^{-1})_{k-1\cdot} - \frac{(A^{-1}x)_{k-1}}{(A^{-1}x)_k}(A^{-1})_{k\cdot} \\ \frac{1}{(A^{-1}x)_k}(A^{-1})_{k\cdot} \\ (A^{-1})_{k+1\cdot} - \frac{(A^{-1}x)_{k+1}}{(A^{-1}x)_k}(A^{-1})_{k\cdot} \\ \vdots \\ (A^{-1})_{n+1\cdot} - \frac{(A^{-1}x)_{n+1}}{(A^{-1}x)_k}(A^{-1})_{k\cdot} \end{pmatrix}.$$

The next lemma describes the cases that can occur when a t-simplex σ in $A(s)$ has at least one s-complete facet τ and a lexicographic pivot rule to $A_{s,\tau}$ is performed when the column to be added corresponds to the vertex of σ opposite τ.

Lemma 3.5

Let σ be a t-simplex of $A(s)$ where s is a sign vector with t zero components. If σ has an s-complete facet τ, then exactly one of the following cases holds:

1. *The simplex σ is an $\tilde{s}$-complete simplex in $A(\tilde{s})$ for precisely one sign vector $\tilde{s}$ with $|I^0(\tilde{s})| = t+1$ and no facets of σ other than τ are s-complete;*
2. *The simplex σ has exactly one other s-complete facet $\tilde{\tau}$ and σ is not an $\tilde{s}$-complete simplex in $A(\tilde{s})$ for any sign vector $\tilde{s}$ with $|I^0(\tilde{s})| = t+1$.*

Proof

Let x^{t+1} be the vertex of σ not contained in τ, and let $y = A_{s,\tau}^{-1}(1, Z(x^{t+1})^\top)^\top$. Since $A_{s,\tau}A_{s,\tau}^{-1} = E^{n+1}$ it holds that $\sum_{i=1}^{t}(A_{s,\tau}^{-1})_{i1} = 1$ and $\sum_{i=1}^{t}(A_{s,\tau}^{-1})_{ij} = 0$, $\forall j \in I_{n+1} \setminus \{1\}$. Suppose that the first n components of y are non-positive. Then $0 \geq (1^{t^\top}, 0^{n-t+1^\top})y = (1^{t^\top}, 0^{n-t+1^\top})A_{s,\tau}^{-1}(1, Z(x^{t+1})^\top)^\top = (1, 0^{n^\top})(1, Z(x^{t+1})^\top)^\top = 1$. This is a contradiction. So it is possible to choose $k \in I_n$ such that $\frac{1}{y_k}(A_{s,\tau}^{-1})_{k\cdot}$ is minimal according to the lexicographic ordering over all row vectors $\frac{1}{y_j}(A_{s,\tau}^{-1})_{j\cdot}$ for which $y_j > 0$ and $j \in I_n$. It is clear that k is uniquely determined, since otherwise $A_{s,\tau}^{-1}$ would be singular. Now it holds that either $k \in I_n \setminus I_t$ or $k \in I_t$.

If $k \in I_n \setminus I_t$ then let $\tilde{s}_{i_s^{k-t}} = 0$, and $\tilde{s}_j = s_j$ for $j \in I_n \setminus \{i_s^{k-t}\}$. Clearly $\sigma \subset A(\tilde{s})$ and $|I^0(\tilde{s})| - 1$ equals the dimension of σ. The matrix $A_{\tilde{s},\sigma}$ is obtained by replacing column k of $A_{s,\tau}$ by the vector $(1, Z(x^{t+1})^\top)^\top$ and interchanging columns $t+1$ and k. Using Lemma 3.4 $A_{\tilde{s},\sigma}^{-1}$ is semi-lexicopositive. So σ is an $\tilde{s}$-complete t-simplex in $A(\tilde{s})$.

If $k \in I_t$ then let $\tilde{\tau}$ be the facet of σ whose vertices are the vertices of σ opposite the vertex x^k. Using Lemma 3.4 and the choice of k it follows that $A_{s,\tilde{\tau}}$ is semi-lexicopositive and hence $\tilde{\tau}$ is an s-complete simplex in $A(s)$.

It follows directly from Lemma 3.4 that if some column $j \in I_n$ of $A_{s,\tau}$ other than column k is replaced, then the inverse of the new matrix is not semi-lexicopositive.

Q.E.D.

The next lemma gives the cases that can occur when an s-complete $(t-1)$-simplex τ in $A(s)$ lies in a $(t-1)$-dimensional set $A(\tilde{s})$ and a lexicographic pivot rule to $A_{s,\tau}$ is performed when the column to be added is equal to $(0, \tilde{s}_{\tilde{k}}e^{\tilde{k}^\top})^\top$ for the unique element $\tilde{k}$ in the set $(I^-(\tilde{s}) \cup I^+(\tilde{s})) \setminus (I^-(s) \cup I^+(s))$.

Lemma 3.6

Let τ be an s-complete $(t-1)$-simplex of $A(s)$ where s is a sign vector with t zero components. Suppose that τ lies in $A(\tilde{s})$ where $\tilde{s}$ has $t-1$ zero components. Then exactly one of the following cases holds:

1. *The simplex τ is equal to $\{0^n\}$ or equal to $\{1^n\}$;*
2. *The simplex τ is an $\bar{s}$-complete simplex in $A(\bar{s})$ for precisely one sign vector $\bar{s} \neq s$ with $|I^0(\bar{s})| = t$ and τ has no $\tilde{s}$-complete facets;*
3. *Precisely one facet of τ is $\tilde{s}$-complete and τ is not an $\bar{s}$-complete simplex in $A(\bar{s})$ for any sign vector $\bar{s} \neq s$.*

Proof

For some unique index $\tilde{k} \in I_n$, $s_{\tilde{k}} = 0$ and $\tilde{s}_{\tilde{k}} \neq 0$. Let $y = A_{s,\tau}^{-1}(0, \tilde{s}_{\tilde{k}} e^{\tilde{k}^\top})^\top$. Exactly one of the following three possibilities occurs: (i) $\tilde{s}_{\tilde{k}} = +1$ and $y_j \leq 0$, $\forall j \in I_n$; (ii) $\tilde{s}_{\tilde{k}} = -1$ and $y_j \leq 0$, $\forall j \in I_n$; (iii) $\exists j \in I_n$ such that $y_j > 0$.
If $\tilde{s}_{\tilde{k}} = +1$ and $y_j \leq 0$, $\forall j \in I_n$, then since $A_{s,\tau} y = (0, \tilde{s}_{\tilde{k}} e^{\tilde{k}^\top})^\top$ we have $y_1 = \cdots = y_t = 0$. So $1 = \sum_{j=1}^{n+1} (A_{s,\tau})_{\tilde{k}+1,j} y_j = \sum_{j=t+1}^{n+1} (A_{s,\tau})_{\tilde{k}+1,j} y_j = -y_{n+1}$, where for the last equality it is used that $s_{\tilde{k}} = 0$. Hence $y_{n+1} = -1$ and consequently

$$\sum_{j=t+1}^{n} (A_{s,\tau})_{\cdot j} y_j = \sum_{j=1}^{n-t} s_{i_s^j} e^{i_s^j} y_{t+j} = (0, e^{\tilde{k}^\top})^\top - (0, 1^{n^\top})^\top.$$

Therefore $t = 1$ and $s_{i_s^j} = +1$, $\forall j \in I_{n-t}$. So $\tilde{s} = +1^n$ and therefore $\tau = \{0^n\}$.
If $\tilde{s}_{\tilde{k}} = -1$ and $y_j \leq 0$, $\forall j \in I_n$, then again $y_1 = \cdots = y_t = 0$. So $-1 = \sum_{j=t+1}^{n+1} (A_{s,\tau})_{\tilde{k}+1,j} y_j = -y_{n+1}$, and hence $y_{n+1} = 1$. Consequently

$$\sum_{j=t+1}^{n} (A_{s,\tau})_{\cdot j} y_j = \sum_{j=1}^{n-t} s_{i_s^j} e^{i_s^j} y_{t+j} = (0, 1^{n^\top})^\top - (0, e^{\tilde{k}^\top})^\top.$$

Therefore $t = 1$ and $s_{i_s^j} = -1$, $\forall j \in I_{n-t}$. So $\tilde{s} = -1^n$ and therefore $\tau = \{1^n\}$.
If there exists some $j \in I_n$ such that $y_j > 0$ then it is possible to choose $k \in I_n$ as in the proof of Lemma 3.5. Again, either $k \in I_n \setminus I_t$ or $k \in I_t$.
If $k \in I_n \setminus I_t$, let $\bar{s}_{\tilde{k}} = \tilde{s}_{\tilde{k}}$, $\bar{s}_{i_s^{k-t}} = 0$, and $\bar{s}_j = \tilde{s}_j$, $\forall j \in I_n \setminus \{i_s^{k-t}, \tilde{k}\}$, and consider $A_{\bar{s},\tau}$. Using Lemma 3.4, the choice of k guarantees that $A_{\bar{s},\tau}^{-1}$ is semi-lexicopositive and therefore τ is $\bar{s}$-complete in $A(\bar{s})$.
If $k \in I_t$, let τ' be the facet of τ opposite to the vertex x^k. By Lemma 3.4 and the choice of k, $A_{\tilde{s},\tau'}^{-1}$ is semi-lexicopositive and hence τ' is $\tilde{s}$-complete in $A(\tilde{s})$.
It follows directly from Lemma 3.4 that if some column $j \in I_n$ of $A_{s,\tau}$ other than column k is replaced, the inverse of the new matrix is not semi-lexicopositive.

Q.E.D.

An s^1-complete simplex τ^1 in $A(s^1)$ and an s^2-complete simplex τ^2 in $A(s^2)$ are said to be adjacent complete simplices if $s^1 = s^2 = s$ and τ^1 and τ^2 are both facets of a simplex σ in $A(s)$, or if τ^1 is a facet of τ^2 and τ^2 is a simplex in $A(s^1)$, or if τ^2 is a facet of τ^1 and τ^1 is a simplex in $A(s^2)$. Using the lemmas above it can be shown that there exists a finite sequence of adjacent complete simplices connecting the simplices $\{0^n\}$ and $\{1^n\}$.

Theorem 3.7

Let $\mathcal{T}$ be a triangulation of C^n, let $\zeta : C^n \to \mathbf{R}^n$ satisfy Condition B, and let $Z : C^n \to \mathbf{R}^n$ be a piecewise linear approximation of ζ with respect to $\mathcal{T}$ which is chosen such that it satisfies Condition C. Let τ be an s-complete simplex in $A(s)$, which is a face of a simplex in $\mathcal{T}$. If $\tau = \{0^n\}$ or $\tau = \{1^n\}$ then there exists exactly one adjacent complete simplex to τ, being a face of a simplex in $\mathcal{T}$. Otherwise τ has exactly two adjacent complete simplices, which are faces of simplices in $\mathcal{T}$. Moreover, there exists a finite sequence of simplices $\tau^0, \ldots, \tau^M$ such that $\tau^0 = \{0^n\}$, $\tau^M = \{1^n\}$, τ^{k-1} and τ^k are adjacent complete simplices, and τ^k is a face of a simplex in $\mathcal{T}$, $\forall k \in I_M$.

Proof

By Lemma 3.2, $\tau = \{0^n\}$ is an s-complete simplex in $A(s)$ if and only if $s = s^0$ where $s^0 = (+1^{n-1^\top}, 0)^\top$. Since $\mathcal{T}$ is a triangulation and $\{0^n\}$ is a facet in the boundary of $A(s^0)$, there is a unique 1-simplex σ in $A(s^0)$ such that $\{0^n\}$ is a facet of σ. By Lemma 3.5 either σ is an $\tilde{s}$-complete simplex in $A(\tilde{s})$ for some sign vector $\tilde{s}$ with $|I^0(\tilde{s})| = 2$ and has no other s^0-complete facets, or σ has exactly one other s^0-complete facet and is not an $\tilde{s}$-complete simplex in $A(\tilde{s})$ for any sign vector $\tilde{s}$. Hence there exists exactly one adjacent complete simplex to $\{0^n\}$. The argument for $\tau = \{1^n\}$ is similar.

Now let τ be an s-complete $(t-1)$-simplex in $A(s)$ for some sign vector s where τ is not equal to $\{0^n\}$ or $\{1^n\}$. There are two possibilities, either τ lies in the relative boundary of $A(s)$ or τ lies in the relative interior of $A(s)$.

If τ lies in the relative boundary of $A(s)$ then, by the properties of a triangulation, there is a unique t-simplex σ in $A(s)$ having τ as a facet. By Lemma 3.5 either σ is s'-complete in $A(s')$ for precisely one sign vector s' and has no other s-complete facets, or σ has exactly one other s-complete facet. This yields one adjacent complete simplex to τ. Since τ lies in the relative boundary of $A(s)$, τ lies in $A(\tilde{s})$ for a unique sign vector $\tilde{s}$ with $|I^0(\tilde{s})| = t-1$. By Lemma 3.6 either τ is $\bar{s}$-complete in $A(\bar{s})$ for a unique sign vector $\bar{s} \neq s$ with $|I^0(\bar{s})| = t$ and has no $\tilde{s}$-complete facets, or τ has exactly one $\tilde{s}$-complete facet. In the latter case we are done. In the former case we use the fact that τ is in the relative boundary of $A(\bar{s})$ and hence there is exactly one simplex $\bar{\sigma}$ in $A(\bar{s})$ having τ as a facet. Applying Lemma 3.5 again yields that either $\bar{\sigma}$ is an $\hat{s}$-complete simplex in $A(\hat{s})$ and has no other $\bar{s}$-complete facets, or $\bar{\sigma}$ has exactly one other $\bar{s}$-complete facet. This shows that τ has exactly two adjacent complete simplices.

If τ lies in the relative interior of $A(s)$ then by the properties of a triangulation, it holds that τ is a facet of exactly two simplices in $A(s)$. Applying Lemma 3.5 twice shows that τ has exactly two adjacent complete simplices.
Now let $\tau^0 = \{0^n\}$. Let τ^1 be the unique adjacent complete simplex to τ^0. Given τ^k for some $k \geq 1$ which is not equal to $\{1^n\}$ there exists a unique adjacent complete simplex τ^{k+1} which is not equal to τ^{k-1}. Now the "door-in door-out" argument of Lemke and Howson (1964) excludes cycling and the finiteness of the number of faces of the simplices in the triangulation $\mathcal{T}$ yields that in a finite number of steps, say M, the simplex τ^M must be equal to $\{1^n\}$. This simplex has no other adjacent complete simplex but τ^{M-1}.

Q.E.D.

We summarize the results above by remarking that it is possible to generate the simplices $\tau^0, \ldots, \tau^M$ given in Theorem 3.7 by the following algorithm.

<u>**Algorithm**</u>

Step 0. Let $i = 0, t = 1, x^1 = 0^n, \tau^i = \tau(x^1), s = (+1^{n-1^\top}, 0)^\top$, and let x^{t+1} be the unique vertex of the simplex in $A(s)$ containing τ^0 as a facet opposite to it.

Step 1. Let σ be equal to the convex hull of x^{t+1} and τ^i. Pivot $(0, Z(x^{t+1})^\top)^\top$ lexicographically into the linear system corresponding with A_{s,τ^i} yielding as described in Lemma 3.5 a unique column $k \in I_n$ of A_{s,τ^i} which has to be replaced. If $k \in I_n \setminus I_t$ then go to Step 3 with $j' = i_s^{k-t}$.

Step 2. Set $i = i + 1$ and let τ^i be the facet of σ opposite the vertex x^k. If $\tau^i = \{1^n\}$ then the algorithm terminates. If τ^i lies in $A(\tilde{s})$ for some $\tilde{s}$ with $t - 1$ zero components then go to Step 4. Otherwise there is exactly one simplex $\tilde{\sigma}$ in $A(s)$ such that $\tilde{\sigma} \neq \sigma$ and τ^i is a facet of $\tilde{\sigma}$. Go to Step 1 with x^{t+1} as the unique vertex in $\tilde{\sigma}$ opposite to the facet τ^i.

Step 3. Define $\tilde{s}$ by $\tilde{s}_{j'} = 0$ and $\tilde{s}_j = s_j, \ \forall j \in I_n \setminus \{j'\}$. There is a unique simplex $\tilde{\sigma}$ in $A(\tilde{s})$ having σ as a facet. Set $i = i + 1, \ t = t + 1$, and go to Step 1 with x^{t+1} as the unique vertex in $\tilde{\sigma}$ opposite to σ, $s = \tilde{s}$, and $\tau^i = \sigma$.

Step 4. Let σ be equal to τ^i. Pivot $(0, \tilde{s}_{\tilde{k}} e^{\tilde{k}^\top})^\top$ lexicographically into the linear system determined by A_{s,τ^i} where $\tilde{k} \in I_n$ is chosen such that $s_{\tilde{k}} = 0$ and $\tilde{s}_{\tilde{k}} \neq 0$. By Lemma 3.6 there is a unique column $k \in I_n$ of A_{s,τ^i} which has to be replaced. If $k \in I_n \setminus I_t$ then go to Step 3 with $j' = i_s^{k-t}, s = \tilde{s}, t = t - 1$, and $i = i - 1$. Otherwise go to Step 2 with $s = \tilde{s}$ and $t = t - 1$.

It is worthwile to mention that it is also possible to start the algorithm with the simplex $\{1^n\}$ and terminate with the simplex $\{0^n\}$. In Theorem 3.8 it is shown that the sequence of adjacent simplices generated by the algorithm yields a piecewise linear path of points in C^n connecting 0^n and 1^n such that for every point x on the path it holds that $0 \le Z_j(x) \le \beta$ if $x_j = 0$, $Z_j(x) = \beta$ if $0 < x_j < 1$, and $0 \ge Z_j(x) \ge \beta$ if $x_j = 1$.

Theorem 3.8

Let $\mathcal{T}$ be a triangulation of C^n and let the correspondence $\zeta : C^n \to \mathbf{R}^n$ satisfy Condition B. Let $Z : C^n \to \mathbf{R}^n$ be a piecewise linear approximation of ζ with respect to $\mathcal{T}$ satisfying Condition C. Then there exists a continuous, piecewise linear function $f : [0,1] \to C^n$ satisfying $f(0) = 0^n$, $f(1) = 1^n$, and $\forall r \in [0,1]$ it holds that for some $\beta \in \mathbf{R}$

$$\begin{aligned} 0 \le Z_j(f(r)) &\le \beta \text{ if } f_j(r) = 0, \\ Z_j(f(r)) &= \beta \text{ if } 0 < f_j(r) < 1, \\ 0 \ge Z_j(f(r)) &\ge \beta \text{ if } f_j(r) = 1. \end{aligned}$$

Proof

Consider all different pairs of sign vectors and simplices $(s^0, \tau^0), \ldots, (s^{M'}, \tau^{M'})$ subsequently generated by the algorithm, satisfying τ^k is s^k-complete in $A(s^k)$, $\forall k \in \{0\} \cup I_{M'}$, and let $t^k = |I^0(s^k)|$. Notice that $M' \ge M$, where M is as in Theorem 3.7, with $M' > M$ only if Case 2 of Lemma 3.6, which corresponds with the first case in Step 4 of the algorithm, occurs during the algorithm. Clearly, $A^{-1}_{s^k,\tau^k}$ is semi-lexicopositive for every $k \in \{0\} \cup I_{M'}$. Consequently $\sum_{j=1}^{n+1} (A_{s^k,\tau^k})_{\cdot j} (A^{-1}_{s^k,\tau^k})_{j1} = (1, 0^{n^\top})^\top$, or equivalently

$$\sum_{j=1}^{t^k} (A^{-1}_{s^k,\tau^k})_{j1} \begin{pmatrix} 1 \\ Z(x^j) \end{pmatrix} + \sum_{j=1}^{n-t^k} (A^{-1}_{s^k,\tau^k})_{t^k+j,1} \begin{pmatrix} 0 \\ s_{i^j_s} e^{i^j_s} \end{pmatrix} + (A^{-1}_{s^k,\tau^k})_{n+1,1} \begin{pmatrix} 0 \\ -1^n \end{pmatrix} = \begin{pmatrix} 1 \\ 0^n \end{pmatrix},$$

where $(A^{-1}_{s^k,\tau^k})_{j1} \ge 0$, $\forall j \in I_n$. Define $y^k = \sum_{j=1}^{t^k} (A^{-1}_{s^k,\tau^k})_{j1} x^j$ and define $f : [0,1] \to C^n$ by

$$f(r) = (1 - M'r + \lfloor M'r \rfloor) y^{\lfloor M'r \rfloor} + (M'r - \lfloor M'r \rfloor) y^{\lfloor M'r \rfloor + 1}, \ \forall r \in [0,1],$$

where for $x \in \mathbf{R}$, $\lfloor x \rfloor$ denotes the greatest integer which is less than or equal to x, and $y^{M'+1}$ is an arbitrary vector in $\mathbf{R}^n$. This function f satisfies the requirements of the theorem. To verify this, remark that for every $k \in I_{M'}$, $A_{s^{k-1},\tau^{k-1}}$ and A_{s^k,τ^k} have n columns in common. Let $B_{\bar{s}^{k-1},\sigma^{k-1}}$ be the $(n+1) \times (n+2)$ matrix which contains all the columns of $A_{s^{k-1},\tau^{k-1}}$ and A_{s^k,τ^k}, $\forall k \in I_{M'}$. The matrix $B_{\bar{s}^{k-1},\sigma^{k-1}}$ yields the system as in (1) for a simplex σ^{k-1} which is the convex hull of τ^{k-1} and τ^k and a sign vector $\bar{s}^{k-1}$ which is such that $I^-(\bar{s}^{k-1}) = I^-(s^{k-1}) \cup I^-(s^k)$ and $I^+(\bar{s}^{k-1}) = I^+(s^{k-1}) \cup I^+(s^k)$. It is easily verified that σ^{k-1} is a simplex in $A(\bar{s}^{k-1})$. The first columns of both $A^{-1}_{s^{k-1},\tau^{k-1}}$ and $A^{-1}_{s^k,\tau^k}$ extended with a zero component, yield admissible solutions to the system in (1) induced by $B_{\bar{s}^{k-1},\sigma^{k-1}}$.

Finally it has to be remarked that if $\tilde{y}, \bar{y} \in \mathbf{R}^{n+2}$ both are admissible solutions to the system $B_{\bar{s}^{k-1},\sigma^{k-1}} y = (1, 0^{n^\top})^\top$ then $\lambda\tilde{y} + (1-\lambda)\bar{y}$ is also an admissible solution for every $\lambda \in [0,1]$. Since for every $k \in I_{M'}$ the simplex σ^{k-1} lies in $A(\bar{s}^{k-1})$ it follows from (1) that for every $r \in [0,1]$ it holds that $0 \leq Z_j(f(r)) \leq \beta$ if $f_j(r) = 0$, $Z_j(f(r)) = \beta$ if $0 < f_j(r) < 1$, and $0 \geq Z_j(f(r)) \geq \beta$ if $f_j(r) = 1$.

Q.E.D.

4 Accuracy Analysis

First it will be argued that the points lying on the path given in Theorem 3.8 indeed all correspond to approximate zero points. To show this, a sequence of triangulations $\mathcal{T}^r$ with mesh size converging to zero is taken. This yields according to Theorem 3.8 for every $r \in \mathbf{N}$ a continuous piecewise linear function $f^r : [0,1] \to C^n$ with image set $f^r([0,1])$ connecting 0^n and 1^n. It will be shown that if q^r is an arbitrary point in $f^r([0,1])$ and the sequence $(q^r)_{r\in\mathbf{N}}$ converges to q then $0^n \in \zeta(q)$. It should be remarked that by the compactness of C^n every sequence of points in C^n has a converging subsequence. From this one obtains that for $\varepsilon > 0$ there exists an $R \in \mathbf{N}$ such that for all $r \geq R$ it holds that $q^r \in f^r([0,1])$ implies $\|Z^r(q^r)\|_\infty < \varepsilon$, or equivalently $\max_{q^r \in f^r([0,1])} \|Z^r(q^r)\|_\infty \to 0$, where $Z^r : C^n \to \mathbf{R}^n$ is the piecewise linear approximation of ζ with respect to $\mathcal{T}^r$. Finally it will be shown that there is a continuum of zero points of ζ being approximated.

Theorem 4.1
Let $\zeta : C^n \to \mathbf{R}^n$ be a correspondence satisfying Condition B. For $r \in \mathbf{N}$ let $\mathcal{T}^r$ be a triangulation of C^n with mesh size smaller than $\frac{1}{r}$, and let $Z^r : C^n \to \mathbf{R}^n$ be a piecewise linear approximation of ζ with respect to $\mathcal{T}^r$, satisfying Condition C. Let $(q^r)_{r\in\mathbf{N}}$ be an arbitrary convergent sequence of points in C^n with limit q^* such that $q^r \in f^r([0,1])$, where f^r corresponds with the function f of Theorem 3.8 for the piecewise linear approximation Z^r. Then $0^n \in \zeta(q^*)$.

Proof
Let $(\lambda_1^r, \ldots, \lambda_{n+1}^r, x^{1^r}, \ldots, x^{n+1^r}, z^{1^r}, \ldots, z^{n+1^r})_{r\in\mathbf{N}}$ be a sequence of points in $\prod_{i=1}^{n+1}[0,1] \times \prod_{i=1}^{n+1} C^n \times \prod_{i=1}^{n+1} \mathbf{R}^n$ satisfying $\sum_{j=1}^{n+1} \lambda_j^r = 1$, $\sigma(x^{1^r}, \ldots, x^{n+1^r})$ is a simplex of $\mathcal{T}^r$, $q^r = \sum_{j=1}^{n+1} \lambda_j^r x^{j^r}$, and $z^{j^r} = Z^r(x^{j^r})$. Notice that it may happen that $\lambda_j^r = 0$ for some $j \in I_{n+1}$. By definition, $Z^r(q^r) = \sum_{j=1}^{n+1} \lambda_j^r z^{j^r}$. Define $z^r = Z^r(q^r)$, then $z^r = \beta^r 1^n - \sum_{j\in\{k\in I_n | q_k^r = 0\}} \mu_j^r e^j + \sum_{j\in\{k\in I_n | q_k^r = 1\}} \mu_j^r e^j$, where $\mu_j^r \geq 0$, $\forall j \in \{k \in I_n \mid q_k^r = 0 \text{ or } q_k^r = 1\}$. Since $\cup_{q\in C^n}\zeta(q)$ is bounded, the sequence given above remains in a compact set, and without loss of generality it can be assumed to converge to an element $(\lambda_1^*, \ldots, \lambda_{n+1}^*, x^{*1}, \ldots, x^{*n+1}, z^{*1}, \ldots, z^{*n+1})$. Define $z^* = \sum_{j=1}^{n+1} \lambda_j^* z^{*j}$. Clearly it holds that $z^r \to z^*$. Since for every $r \in \mathbf{N}$ the mesh size

of T^r is smaller than $\frac{1}{r}$ it holds for every $j \in I_{n+1}$ that $x^{*j} = q^*$. Using that ζ is upper semi-continuous and $\cup_{q \in C^n} \zeta(q)$ is bounded this implies that for every $j \in I_{n+1}$, $z^{*j} \in \zeta(q^*)$. Since ζ is convex-valued, $\sum_{j=1}^{n+1} \lambda_j^* = 1$, and $\lambda_j^* \geq 0$, $\forall j \in I_{n+1}$, it holds that $z^* \in \zeta(q^*)$.
If there is a subsequence $(q^{r^s})_{s \in \mathbf{N}}$ of $(q^r)_{r \in \mathbf{N}}$ such that for every $s \in \mathbf{N}$, $0 < q_j^{r^s} < 1$, $\forall j \in I_n$, then $z^{r^s} = \beta^{r^s} 1^n$, and therefore $z_j^* = z_{j'}^*$, $\forall j, j' \in I_n$. Since $z^* \in \zeta(q^*)$ there is some $p \in \mathbf{R}^n_{++}$ such that $p \cdot z^* = 0$. Consequently $z^* = 0^n$. If there is not such a subsequence, then there exists a subsequence $(q^{r^s})_{s \in \mathbf{N}}$ of $(q^r)_{r \in \mathbf{N}}$ such that for some $k \in I_n$, $\forall s \in \mathbf{N}$, $q_k^{r^s} = 0$, or for some $k \in I_n$, $\forall s \in \mathbf{N}$, $q_k^{r^s} = 1$. In the first case, using that Z^r satisfies Condition C, it holds that $0 \leq z_k^{r^s} \leq \beta^{r^s}$ and therefore $0^n \leq z^{r^s}$, $\forall s \in \mathbf{N}$. This implies that $z^* \geq 0^n$. In the second case, using again that Z^r satisfies Condition C, it holds that $0 \geq z_k^{r^s} \geq \beta^{r^s}$ and therefore $0^n \geq z^*$. In both cases the existence of a $p \in \mathbf{R}^n_{++}$ such that $p \cdot z^* = 0$ implies that $z^* = 0^n$.

Q.E.D.

From Theorem 4.1 the next result immediately follows.

Corollary 4.2
Let $\zeta : C^n \to \mathbf{R}^n$ be a correspondence satisfying Condition B. For $r \in \mathbf{N}$ let T^r be a triangulation of C^n with mesh size smaller than $\frac{1}{r}$, and let $Z^r : C^n \to \mathbf{R}^n$ be a piecewise linear approximation of ζ with respect to T^r, satisfying Condition C. Then for any $\varepsilon > 0$ there exists an $R \in \mathbf{N}$ such that for every $r \geq R$ it holds that $q^r \in f^r([0,1])$ implies $\|Z^r(q^r)\|_\infty < \varepsilon$.

Proof
Suppose a sequence $(q^r, Z^r(q^r))_{r \in \mathbf{N}}$ exists with $q^r \in f^r([0,1])$ and $\|Z^r(q^r)\|_\infty \geq \varepsilon$ for every $r \in \mathbf{N}$. Since C^n is compact and $\cup_{q \in C^n} \zeta(q)$ is bounded there exists a converging subsequence $(q^{r^s}, Z^{r^s}(q^{r^s}))_{s \in \mathbf{N}}$, with limit say (q^*, z^*), where $\|z^*\|_\infty \geq \varepsilon > 0$. As in the proof of Theorem 4.1 it can be shown that $z^* = 0^n$, yielding a contradiction.

Q.E.D.

Using Theorem 4.1 it is easily shown that there exists a continuum of points $q \in C^n$ such that $0^n \in \zeta(q)$. Intuitively, a continuum is a set having the same "number" of elements as the unit interval $(0,1)$. We will make this statement more rigorous. Two sets S and T are said to be equivalent, denoted by $S \sim T$, if there exists a function $g : S \to T$, not necessarily continuous, which is bijective. Now a continuum is a set equivalent to the unit interval $(0,1)$. It can be shown that the unit cube C^k is a continuum for every $k \in \mathbf{N}$, see for example Williamson (1962, p.5). In order to show that a particular set is a continuum the following theorem, e.g. shown in Williamson (1962, p.5), is very useful.

Theorem 4.3 (Bernstein)
Let be given the sets S^1, S^2, and S^3. If $S^1 \subset S^2 \subset S^3$ and $S^1 \sim S^3$ then $S^2 \sim S^3$.

Consider again a sequence of triangulations $(\mathcal{T}^r)_{r\in\mathbf{N}}$ of C^n with mesh size converging to zero, let Z^r be a piecewise linear approximation of ζ with respect to $\mathcal{T}^r$, and let the function f^r be given as in Theorem 3.8. Moreover, let some constant $\alpha \in [0,1]$ be given. Since f^r is a continuous function, $f^r(0) = 0^n$, and $f^r(1) = 1^n$, there exists a $t^r \in [0,1]$ such that $f^r_1(t^r) = \alpha$. Now define the sequence $(q^r)_{r\in\mathbf{N}}$ by $q^r = f^r(t^r)$. Without loss of generality it can be assumed that $q^r \to q$, for some $q \in C^n$. By Theorem 4.1 it holds that $0^n \in \zeta(q)$. Clearly $q_1 = \alpha$. Hence for every $\alpha \in [0,1]$ there exists a vector $q^\alpha \in C^n$ such that $q^\alpha_1 = \alpha$ and $0^n \in \zeta(q^\alpha)$. Since $\{q^\alpha \in C^n \mid \alpha \in [0,1]\} \subset \{q \in C^n \mid 0^n \in \zeta(q)\} \subset C^n$ and both the first and last set is a continuum our statement is shown using Theorem 4.3.

Now consider again the case where the correspondence ζ is the excess demand correspondence of an economy with price rigidities $E = (\{X^i, \succeq^i, w^i\}_{i=1}^m, P)$ and rationing system $(\hat{l}, \hat{L})$. If $\underline{p}_1 = \bar{p}_1 = 1$ we obtain the model of Drèze (1975) where commodity 1 denotes the numeraire commodity. His equilibrium concept, the so-called Drèze equilibrium, corresponds to an element $q \in C^n$ such that $0^n \in \zeta(q)$ and $q_1 = \frac{1}{2}$. The Drèze equilibrium is therefore a constrained equilibrium without rationing on the market of the numeraire commodity. By taking $\alpha = \frac{1}{2}$ the previous paragraph shows that on the path generated by the algorithm there is an approximate Drèze equilibrium. Another equilibrium concept is given in van der Laan (1980). His equilibrium concept, the so-called supply constrained equilibrium, corresponds to an element $q \in C^n$ such that $0^n \in \zeta(q)$ and $\max_{j\in I_n} q_j = \frac{1}{2}$. The supply constrained equilibrium is therefore a constrained equilibrium without rationing on the market of at least one commodity, and without demand rationing on every market. We claim that there is an approximate supply constrained equilibrium on the path generated by the algorithm.

Consider a sequence of triangulations $(\mathcal{T}^r)_{r\in\mathbf{N}}$ of C^n with mesh size converging to zero with piecewise linear approximation Z^r with respect to $\mathcal{T}^r$ of ζ and the function f^r given in Theorem 3.8. Consider the set $S = \{q \in C^n \mid \max_{j\in I_n} q_j = \frac{1}{2}\}$. The element $q \in C^n$ corresponds to a supply constrained equilibrium if $q \in S$ and $0^n \in \zeta(q)$. Since f^r is a continuous function, $f^r(0) = 0^n$ and $f^r(1) = 1^n$, there exists a $t^r \in [0,1]$ such that $f^r(t^r) \in S$. Now define the sequence $(q^r)_{r\in\mathbf{N}}$ by $q^r = f^r(t^r)$. Without loss of generality it can be assumed that $q^r \to q$. By Theorem 4.1 it holds that $0^n \in \zeta(q)$. Obviously $q \in S$.

For the rest of this section, consider the case where ζ satisfying Condition B is a continuous function, denoted by z. Then there exists a function $p : C^n \to \mathbf{R}^n_{++}$ (not necessarily continuous) satisfying $p^\top(q) \cdot z(q) = 0$. We will now derive the accuracy of solutions generated by the algorithm. Let $\varepsilon > 0$ be given and let $\delta > 0$ be such that for $\tilde{q}, \bar{q} \in Q$, $\|\tilde{q} - \bar{q}\|_\infty < \delta$ implies $\|z(\tilde{q}) - z(\bar{q})\|_\infty < \varepsilon$. By the continuity of z and the compactness of C^n such a δ exists. Consider a triangulation $\mathcal{T}$ with mesh size less than or equal to δ and suppose q lies in the image set $f([0,1])$ of the piecewise linear function f given by Theorem 3.8. Then there exist numbers $\lambda_j \geq 0$, $\forall j \in I_{n+1}$, $\mu_j \geq 0$, $\forall j \in I_n$, and $\beta \in \mathbf{R}$ such that $\sum_{j=1}^{n+1} \lambda_j = 1$, $q = \sum_{j=1}^{n+1} \lambda_j x^j$ for vertices x^j of a simplex in $\mathcal{T}$ containing q, and for every $j \in I_n$

$$\begin{aligned} Z_j(q) &= \beta - \mu_j \text{ if } q_j = 0, \\ Z_j(q) &= \beta \text{ if } 0 < q_j < 1, \\ Z_j(q) &= \beta + \mu_j \text{ if } q_j = 1. \end{aligned}$$

Clearly $\|Z(q) - z(q)\|_\infty = \|\sum_{j=1}^{n+1} \lambda_j(z(x^j) - z(q))\|_\infty \leq \varepsilon$. Hence $Z(q) - \varepsilon 1^n \leq z(q) \leq Z(q) + \varepsilon 1^n$. If for some $j \in I_n$, $q_j = 0$ then $\beta - \mu_j = Z_j(q) \geq 0$, so $\beta \geq \mu_j \geq 0$. If for some $j \in I_n$, $q_j = 1$ then $\beta + \mu_j = Z_j(q) \leq 0$, so $\beta \leq -\mu_j \leq 0$.

Consider the case where for some $j \in I_n$, $q_j = 0$, and for some $j \in I_n$, $q_j = 1$. Then $\beta = 0$ and $\mu_j = 0$, $\forall j \in I_n$ satisfying $q_j = 0$ or $q_j = 1$. So $-\varepsilon 1^n \leq z(q) \leq \varepsilon 1^n$. Moreover, if $q_j = 0$ then $0 \leq z_j(q) \leq \varepsilon$, and if $q_j = 1$ then $-\varepsilon \leq z_j(q) \leq 0$.

Consider the case where for every $j \in I_n$, $0 < q_j < 1$. So $Z(q) = \beta 1^n$. Then

$$|\beta| \sum_{j=1}^{n} p_j(q) = |p^\top(q) Z(q)| = |p^\top(q)(Z(q) - z(q))| \leq \varepsilon \sum_{j=1}^{n} p_j(q).$$

So $|\beta| \leq \varepsilon$. Consequently $(\beta - \varepsilon)1^n \leq z(q) \leq (\beta + \varepsilon)1^n$ with $|\beta| < \varepsilon$.

Next consider the case where for some $j \in I_n$, $q_j = 1$, and for every $j \in I_n$, $q_j > 0$. Since $\beta + \mu_j \leq 0$ if $q_j = 1$ it holds that $\beta \leq 0$ and $0 \leq \mu_j \leq -\beta$. Moreover,

$$0 = p^\top(q) z(q) \leq (\beta + \varepsilon) \sum_{j \in \{k \in I_n | 0 < q_k < 1\}} p_j(q),$$

implies $\beta + \varepsilon \geq 0$ and therefore $-\varepsilon \leq \beta \leq 0$. Hence

$$\begin{aligned} &\beta - \varepsilon \leq z_j(q) \leq 0 \text{ if } q_j = 1, \\ &\beta - \varepsilon \leq z_j(q) \leq \beta + \varepsilon \text{ if } 0 < q_j < 1, \end{aligned}$$

with $-\varepsilon \leq \beta \leq 0$.

Finally, consider the case where for some $j \in I_n$, $q_j = 0$, and for every $j \in I_n$, $q_j < 1$. Similarly as in the previous paragraph it can be shown that

$$\begin{aligned} &0 \leq z_j(q) \leq \beta + \varepsilon \text{ if } q_j = 0, \\ &\beta - \varepsilon \leq z_j(q) \leq \beta + \varepsilon \text{ if } 0 < q_j < 1, \end{aligned}$$

with $0 \leq \beta \leq \varepsilon$.

All cases are summarized in Theorem 4.4.

Theorem 4.4
Let $z : C^n \to \mathbf{R}^n$ be a function satisfying Condition B. Let $\varepsilon > 0$ be given and choose $\delta > 0$ such that for all $\tilde{q}, \bar{q} \in C^n$, $\|\tilde{q} - \bar{q}\|_\infty < \delta$ implies $\|z(\tilde{q}) - z(\bar{q})\|_\infty < \varepsilon$. Let $\mathcal{T}$ be a triangulation with mesh size less than δ and let $q \in f([0,1])$ with f as in Theorem 3.8. Then there is a $\beta \in \mathbf{R}$ satisfying $-\varepsilon \leq \beta \leq \varepsilon$, $0 \leq \beta$ if $q_j = 0$ for some $j \in I_n$, and $\beta \leq 0$ if $q_j = 1$ for some $j \in I_n$, such that

$$\begin{aligned}
&0 \leq z_j(q) \leq \beta + \varepsilon \text{ if } q_j = 0,\\
&\beta - \varepsilon \leq z_j(q) \leq \beta + \varepsilon \text{ if } 0 < q_j < 1,\\
&\beta - \varepsilon \leq z_j(q) \leq 0 \text{ if } q_j = 1.
\end{aligned}$$

5 An Illustration of the Algorithm

In order to illustrate the algorithm we give an example in this section of a correspondence ζ satisfying Condition B. The example will show some interesting features of the algorithm. We choose the example such that it is possible to determine analytically all elements $q \in C^n$ satisfying $\zeta(q) = 0^n$, and therefore it is possible to compare the set of approximate zero points generated by the algorithm with the exact zero points. The correspondence ζ in the example is the excess demand correspondence of an economy with price rigidities, $E = (\{X^i, \succeq^i, w^i\}_{i=1}^2, P)$, where $X^1 = X^2 = \mathbf{R}^2_+$, $\succeq^1$ and $\succeq^2$ can be represented by utility functions defined by $u^1(x_1, x_2) = (x_1)^{\frac{3}{4}}(x_2)^{\frac{1}{4}}$, $\forall x \in \mathbf{R}^2_+$, and $u^2(x_1, x_2) = (x_1)^{\frac{1}{4}}(x_2)^{\frac{3}{4}}$, $\forall x \in \mathbf{R}^2_+$, respectively, $w^1 = (1,4)^\top$, $w^2 = (2,1)^\top$, and $P = \{p \in \mathbf{R}^2_+ \mid \frac{1}{6} \leq p_1 \leq 2,\ p_2 = 1\}$. The rationing system $(\hat{l}, \hat{L}) : -\mathbf{R}^4_+ \times \mathbf{R}^4_+$ is defined by

$$\begin{aligned}
\hat{l}^1_1(q) = \hat{l}^2_1(q) &= -3\min\{1, 3q_1\},\ \forall q \in C^2,\\
\hat{l}^1_2(q) = \hat{l}^2_2(q) &= -5\min\{1, 3q_2\},\ \forall q \in C^2,\\
\hat{L}^1_1(q) = \hat{L}^2_1(q) &= 18\min\{1, 3 - 3q_1\},\ \forall q \in C^2,\\
\hat{L}^1_2(q) = \hat{L}^2_2(q) &= 5\min\{1, 3 - 3q_2\},\ \forall q \in C^2.
\end{aligned}$$

The function $\hat{p} : C^2 \to \mathbf{R}^2_{++}$ is defined by

$$\begin{aligned}
\hat{p}_1(q) &= \max\{\frac{1}{6}, \min\{\frac{11}{2}q_1 - \frac{5}{3}, 2\}\},\ \forall q \in C^2,\\
\hat{p}_2(q) &= 1,\ \forall q \in C^2.
\end{aligned}$$

It can be easily derived that the excess demand function of consumer 1 is given by

$$z^1(q) = \begin{cases} (90q_2, -15q_1)^\top, & 0 \le q_1 \le \frac{1}{3},\ 0 \le q_2 \le \frac{71}{360}, \\ (\frac{71}{4}, -\frac{71}{24})^\top, & 0 \le q_1 \le \frac{1}{3},\ \frac{71}{360} \le q_2 \le 1, \\ (\frac{90q_2}{33q_1-10}, -15q_2)^\top, & \frac{1}{3} \le q_1 \le \frac{2}{3},\ 0 \le q_2,\ 33q_1 + 360q_2 \le 82, \\ (\frac{82-33q_1}{132q_1-40}, \frac{33q_1-82}{24})^\top, & \frac{1}{3} \le q_1 \le \frac{2}{3},\ q_2 \le 1,\ 33q_1 + 360q_2 \ge 82, \\ (\frac{15q_2}{2}, -15q_2)^\top, & \frac{2}{3} \le q_1,\ 0 \le q_2 \le \frac{1}{6},\ 36q_1 + 5q_2 \le 36, \\ (\frac{5}{4}, -\frac{5}{2})^\top, & \frac{2}{3} \le q_1 \le \frac{211}{216},\ \frac{1}{6} \le q_2 \le 1, \\ (54 - 54q_1, 108q_1 - 108)^\top, & \frac{211}{216} \le q_1 \le 1,\ q_2 \le 1,\ 36q_1 + 5q_2 \ge 36. \end{cases}$$

The excess demand function of consumer 2 is given by

$$z^2(q) = \begin{cases} (0,0)^\top, & 0 \le q_1 \le \frac{1}{3},\ 0 \le q_2 \le 1, \\ (\frac{33-99q_1}{66q_1-20}, \frac{33q_1-11}{4})^\top, & \frac{1}{3} \le q_1 \le \frac{2}{3},\ 0 \le q_2,\ 33q_1 + 60q_2 \le 71, \\ (\frac{90q_2-90}{33q_1-10}, 15 - 15q_2)^\top, & q_1 \le \frac{2}{3},\ q_2 \le 1,\ 33q_1 + 60q_2 \ge 71, \\ (-\frac{11}{8}, \frac{11}{4})^\top, & \frac{2}{3} \le q_1 \le 1,\ 0 \le q_2 \le \frac{49}{60}, \\ (\frac{15q_2-15}{2}, 15 - 15q_2)^\top, & \frac{2}{3} \le q_1 \le 1,\ \frac{49}{60} \le q_2 \le 1. \end{cases}$$

Now the total excess demand correspondence, $z : C^2 \to \mathbf{R}^2$ is given by the function $z = z^1 + z^2$. The zero points of z can easily be determined analytically. It can be verified that the points

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{3} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{148}{231} \\ \frac{71}{420} \end{pmatrix}, \begin{pmatrix} \frac{148}{231} \\ \frac{349}{420} \end{pmatrix}, \begin{pmatrix} \frac{2}{3} \\ \frac{5}{6} \end{pmatrix}, \begin{pmatrix} \frac{211}{216} \\ \frac{5}{6} \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

and the convex combinations of any two subsequent points are all zero points of z. Next we use the algorithm to compute approximate zero points of z. The triangulation used is the K-triangulation, as proposed in Freudenthal (1942). Let $r \in \mathbb{N}$ be given, then the K-triangulation of the cube C^n with grid size r^{-1} is the collection of all simplices $\sigma_{(x^1,\pi)}$ with vertices $x^1, \ldots, x^{n+1}$ in C^n such that each component of x^1 is a multiple of r^{-1}, $\pi = (\pi_1, \ldots, \pi_n)$ is a permutation of the elements of I_n, and for every $j \in I_n$, $x^{j+1} = x^j + r^{-1}e^{\pi_j}$. It can be shown that the K-triangulation of the unit cube indeed satisfies all the conditions required for a triangulation. The mesh size of the K-triangulation of C^n with grid size equal to r^{-1} is equal to $\sqrt{n}r^{-1}$. In Figure I the K-triangulation of C^2 with grid size equal to $\frac{1}{6}$ is drawn. In this figure all adjacent complete 1-simplices are drawn by thick lines. The sequence of adjacent complete simplices $\tau^0, \ldots, \tau^{21}$ corresponds to the one in Theorem 3.8. The piecewise linear path $f([0,1])$ of points generated by the algorithm is given by the points

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{3} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ \frac{11}{120} \end{pmatrix}, \begin{pmatrix} \frac{113}{176} \\ \frac{1}{6} \end{pmatrix}, \begin{pmatrix} \frac{113}{176} \\ \frac{163}{528} \end{pmatrix}, \begin{pmatrix} \frac{277}{429} \\ \frac{1}{3} \end{pmatrix}, \begin{pmatrix} \frac{277}{429} \\ \frac{2}{3} \end{pmatrix}, \begin{pmatrix} \frac{2}{3} \\ \frac{5}{6} \end{pmatrix}, \begin{pmatrix} \frac{5}{6} \\ \frac{5}{6} \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

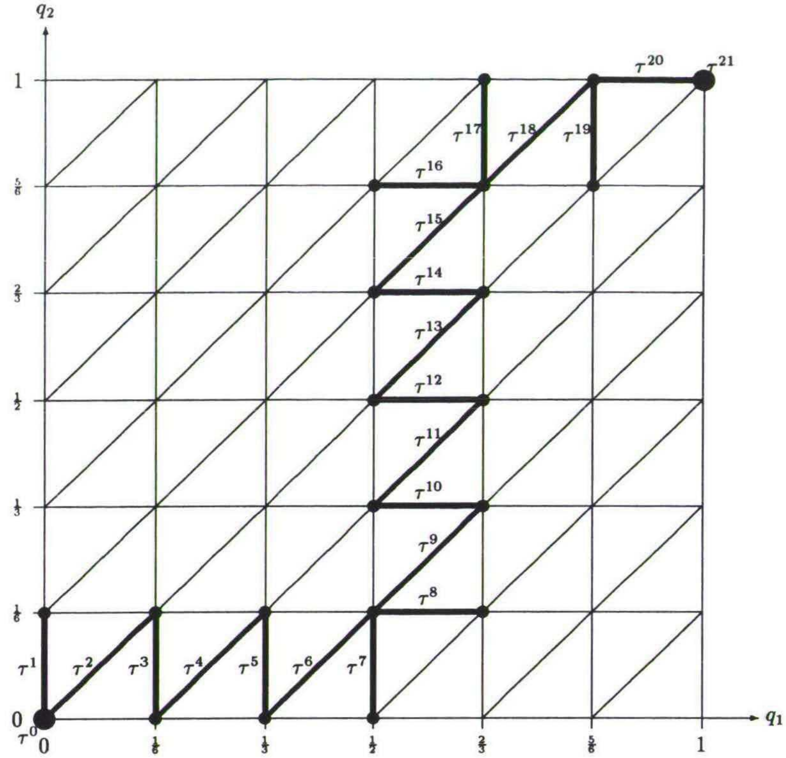

Figure I

and all convex combinations of subsequent points. In Figure II the solid line corresponds to the approximate zero points, while the broken line corresponds to the set of zero points of z. The algorithm starts with the s^0-complete simplex $\tau^0 = \{0^n\}$ in $A(s^0)$, where $s^0 = (+1, 0)^\top$. Notice that although τ^0 is also $(0, -1)^\top$-complete, τ^0 does not lie in $A((0, -1)^\top)$. The simplex τ^0 is not s-complete for $s = (0, +1)^\top$ since A^{-1}_{s,τ^0} is not semi-lexicopositive, although the system $A_{(0,-1)^\top,\tau^0}x = 0$, does have a solution satisfying $x_1 \geq 0,\ x_2 \geq 0$. In this way the modified lexicographic pivot rules determine in a unique way the direction the algorithm will follow. So the next vertex which is brought into the system is $(0, \frac{1}{6})^\top$. Remark that the direction determined by the lexicographic pivot rules is orthogonal to the set of zero points of z around 0^n. It can easily be verified that the 0-simplex $\tau((0, \frac{1}{6})^\top)$ is not $(+1, 0)^\top$-

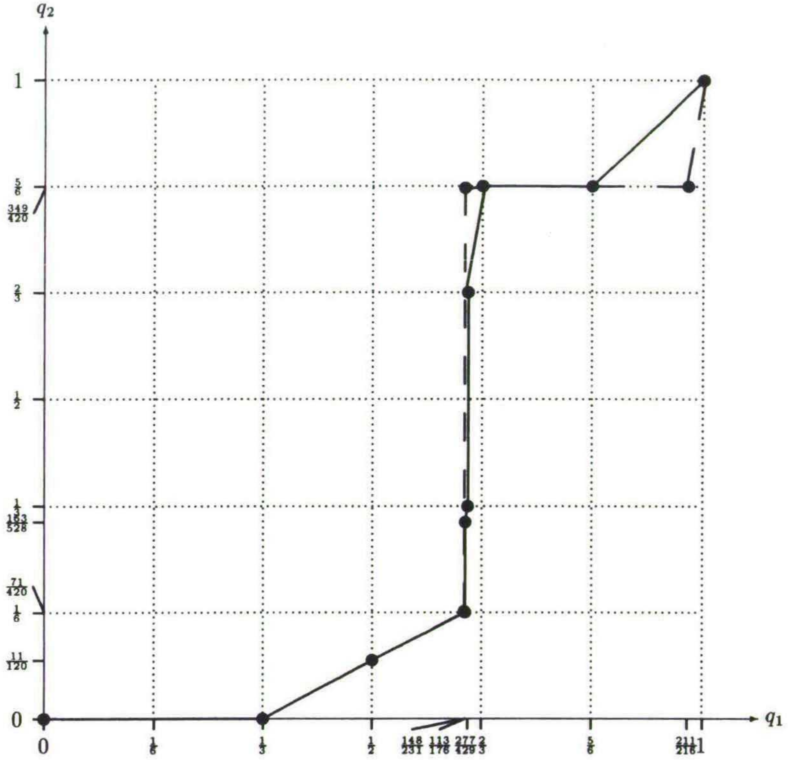

Figure II

complete. However, the 1-simplex $\tau^1 = \tau((0,0)^\top, (0, \frac{1}{6})^\top)$ is $(0,0)^\top$-complete, and therefore we are in Case 1 of Lemma 3.5. The simplex τ^2 is the unique $(0,0)^\top$-complete facet of the simplex $\sigma((0,0)^\top, (0, \frac{1}{6})^\top, (\frac{1}{6}, \frac{1}{6})^\top)$ in $A((0,0)^\top) = C^n$ which is not equal to τ^1. This corresponds to Case 2 of Lemma 3.5.

From Figure II it can be seen that although the lexicographic pivot rules determine an initial direction to leave τ^0 being orthogonal to the set of zero points of z around 0^n, the points on the piecewise linear path of approximate zero points generated by the simplices $\tau^0, \ldots, \tau^5$ are zero points of z. An interesting situation occurs at the simplex

$\tau^{14} = \tau((\frac{1}{2}, \frac{2}{3})^\top, (\frac{2}{3}, \frac{2}{3})^\top)$. For this simplex it holds that

$$A^{-1}_{(0,0)^\top,\tau^{14}} = \begin{pmatrix} \frac{18}{143} & \frac{48}{143} & \frac{-48}{143} \\ \frac{125}{143} & \frac{-48}{143} & \frac{48}{143} \\ \frac{5}{104} & \frac{-7}{13} & \frac{-6}{13} \end{pmatrix}.$$

The vertex brought into the system is given by $x^3 = (\frac{2}{3}, \frac{5}{6})^\top$, with $Z(x^3) = (0,0)^\top$. So $y = A^{-1}_{(0,0)^\top,\tau^{14}}(1, Z(x^3)^\top)^\top = (\frac{18}{143}, \frac{125}{143}, \frac{5}{104})^\top$. Now there occurs a degeneracy problem since both y_1 and y_2 are positive and equal to the corresponding element of the first column of $A^{-1}_{(0,0)^\top,\tau^{14}}$. However, since $\frac{(A^{-1}_{(0,0)^\top,\tau^{14}})_{1\cdot}}{y_1}$ is lexicographically greater than $\frac{(A^{-1}_{(0,0)^\top,\tau^{14}})_{2\cdot}}{y_2}$, the vertex of τ^{14} corresponding with column 2 of $A_{(0,0)^\top,\tau^{14}}$, $(\frac{2}{3}, \frac{2}{3})^\top$, should be replaced by the vertex $(\frac{2}{3}, \frac{5}{6})^\top$, yielding the simplex τ^{15}. Moreover, it has to be remarked that using the usual lexicographic pivoting rules would imply that the last column of $A_{(0,0)^\top,\tau^{14}}$ should be replaced by the vertex $(\frac{2}{3}, \frac{5}{6})^\top$. But then it is impossible to determine a new simplex being $(0,0)^\top$-complete. Another interesting case occurs at τ^{20}. The 1-simplex τ^{20} lies in $A((0,-1)^\top)$ and therefore $(0,0,-1)^\top$ is the new column brought into the system. We are in Case 3 of Lemma 3.6, where τ^{21} is the unique $(0,-1)^\top$-complete facet of τ_{20}. The 0-simplex τ^{21} lies in $A((-1,-1)^\top)$ and we are in Case 1 of Lemma 3.6. The algorithm now terminates with the $(0,-1)^\top$-complete simplex $\{1^n\}$. It is clear from Figure II that the approximate zero points lying in the s-complete $(t-1)$-simplices in $A(s)$ for some sign vector s with $t = |I^0(s)|$ are everywhere extremely close to the zero points of z themselves. It should be remarked that this is caused by the fact that the function z given in the example has such an easy structure. For more complicated correspondences a finer mesh size might be needed to obtain the same accuracy.

The approximate supply constrained equilibrium, being the constrained equilibrium without binding demand rationing and with at least one market without rationing, corresponds with $q^S = (\frac{1}{2}, \frac{11}{120})^\top$ and the total excess demand equals zero in this point, $z(q^S) = (0,0)^\top$. If commodity 2 is assumed to be the numeraire commodity then the approximate Drèze equilibrium, being also the constrained equilibrium without binding supply rationing and with at least one market without rationing, corresponds with $q^D = (\frac{277}{429}, \frac{1}{2})$. The total excess demand in this point is close to zero, $z(q^D) = (-0.0255, 0.0480)^\top$.

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